July 3, 2026

Enriching continuous Lagrange finite element approximation spaces using neural networks

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MS33: Hybrid and Structure Preserving Methods for Scientific Machine Learning

SciCADE 2026
Scientific Computing and Differential Equations

Objective

Develop an hybrid finite element / neural network method.

accurate — quick + parameterized

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Parametric Poisson problem (with homogeneous Dirichlet BC):

For one or several $\boldsymbol{\mu} \in \mathcal{M}$, find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u(\boldsymbol{x};\boldsymbol{\mu}) = f(\boldsymbol{x};\boldsymbol{\mu}), & (\boldsymbol{x},\boldsymbol{\mu}) \in \Omega \times \mathcal{M}, \\ u(\boldsymbol{x};\boldsymbol{\mu}) = 0, & (\boldsymbol{x},\boldsymbol{\mu}) \in \partial\Omega \times \mathcal{M}, \end{cases} \tag{$\mathcal{P}$}$$

with $\Omega \subset \mathbb{R}^d$ a domain ($d = 1, 2, 3$) and $\mathcal{M} \subset \mathbb{R}^p$ the parameter space (p the number of parameters).

Classical approaches

Physics-Informed Neural Networks[1]

Standard PINNs: Find the optimal weights $\theta^\star$, such that

$$\theta^\star = \underset{\theta}{\operatorname{argmin}} \big(\omega_r J_r(\theta) + \omega_b J_b(\theta)\big), \qquad (\mathcal{P}_\theta)$$

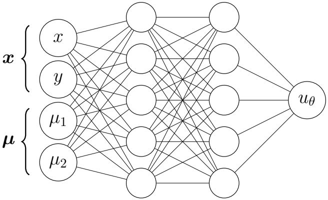
with

residual loss
$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} \big|\Delta u_\theta(\boldsymbol{x}, \boldsymbol{\mu}) + f(\boldsymbol{x}, \boldsymbol{\mu})\big|^2 d\boldsymbol{x} d\boldsymbol{\mu},$$

boundary loss
$$J_b(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega} \big|u_\theta(\boldsymbol{x}, \boldsymbol{\mu})\big|^2 d\boldsymbol{x} d\boldsymbol{\mu},$$

with ω_r and ω_b are some weights.

u_θ is a neural network, e.g. fully-connected NN. (see example in 2D with 2 parameters)



Monte-Carlo method: Discretize the cost functions by random process.

[1] [Raissi, Perdikaris, and Karniadakis, 2019]

Finite Element Method[1]

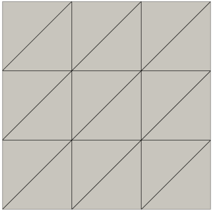
Variational Problem:

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \qquad (\mathcal{P}_h)$$

with h the mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h, \quad l(v_h) = \int_\Omega f v_h,$$

and V_h^0 a continuous, piecewise polynomial space of degree k.



Cartesian mesh

[1] [Ern and Guermond, 2004]

Finite Element Method[1]

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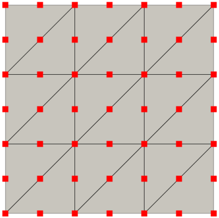
and V_h^0 a continuous, piecewise polynomial space of degree k.

Linear system: Let $(\phi_1, \dots, \phi_{N_{\text{dofs}}})$ a basis of V_h^0.
Find $U \in \mathbb{R}^{N_{\text{dofs}}}$ such that

$$AU = b,$$
$$A = \left(a(\phi_i, \phi_j)\right)_{1 \leq i,j \leq N_{\text{dofs}}} \in \mathbb{R}^{N_{\text{dofs}} \times N_{\text{dofs}}},$$
$$b = \left(l(\phi_j)\right)_{1 \leq j \leq N_{\text{dofs}}} \in \mathbb{R}^{N_{\text{dofs}}}.$$



degrees of freedom ($k = 2$)

[1] [Ern and Guermond, 2004]

Finite Element Method[1]

Variational Problem:

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Find $U \in \mathbb{R}^{N_{\text{dofs}}}$ such that

$$AU = b,$$

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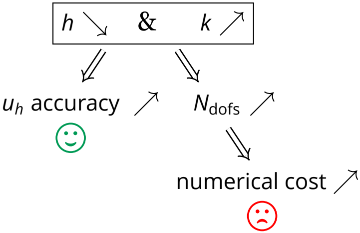


[1] [Ern and Guermond, 2004]

Enriched finite element method using PINNs

Pipeline of the Enriched FEM

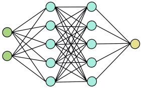
OFFLINE : PINN training

Inputs

$\boldsymbol{x} \in \Omega$: space coordinates

$\boldsymbol{\mu} \in \mathcal{M}$: parameter

$\longrightarrow$



Parametric PINN

$\longrightarrow$

Output

$u_\theta(\boldsymbol{x}, \boldsymbol{\mu})$: prediction of the PDE solution $u(\boldsymbol{x}, \boldsymbol{\mu})$

Pipeline of the Enriched FEM

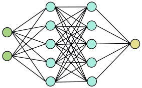
OFFLINE : PINN training

Inputs

$\boldsymbol{x} \in \Omega$: space coordinates

$\boldsymbol{\mu} \in \mathcal{M}$: parameter

→



Parametric PINN

→

Output

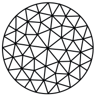
$u_\theta(\boldsymbol{x}, \boldsymbol{\mu})$: prediction of the PDE solution $u(\boldsymbol{x}, \boldsymbol{\mu})$

ONLINE (fixed $\boldsymbol{\mu}$) : PINN evaluation + Enriched FEM resolution

PINN evaluation

$u_\theta(\cdot, \boldsymbol{\mu})$: prediction for the given parameter μ

→



FEM solver enriched by u_θ

→

Output

enriched FEM solution (depending on the mesh size h)

Complete ONLINE process : accurate + quick

Additive approach

Variational Problem: Let $u_\theta \in H^{k+1}(\Omega)$ be a PINN prior.

$$\text{Find } C_h^+ \in V_h \text{ such that, } \forall v_h \in V_h^0, a(C_h^+, v_h) = l(v_h) - a(u_\theta, v_h), \qquad (\mathcal{P}_h^+)$$

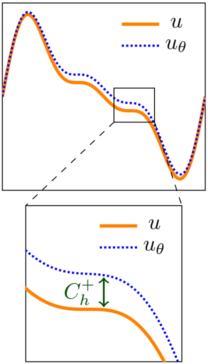
with the enriched trial space V_h^+ defined by

$$V_h^+ = \{u_h^+ = u_\theta + C_h^+, \quad C_h^+ \in V_h^0\}.$$

General Dirichlet BC: If $u = g$ on $\partial\Omega$, then

$$C_h^+ = g - u_\theta \quad \text{on } \partial\Omega,$$

with u_θ the PINN prior.



We expect that the modified problem will give the same results as the standard one on coarser meshes.

Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

Let $u_h \in V_h$ be the solution of $(\mathcal{P}_h)$ with V_h the standard trial space. Then,

$$\|u - u_h\|_{L^2} \leqslant C_{L^2}\, h^{k+1} |u|_{H^{k+1}} .$$

Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

Let $u_h \in V_h$ be the solution of $(\mathcal{P}_h)$ with V_h the standard trial space. Then,

$$\|u - u_h\|_{L^2} \leqslant C_{L^2}\, h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [Lecourtier et al., 2026]

Let $u_h^+ \in V_h^+$ be the solution of $(\mathcal{P}_h^+)$ with V_h^+ the enriched trial space. Then,

$$\|u - u_h^+\|_{L^2} \leqslant \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} \left(C_{L^2}\, h^{k+1} |u|_{H^{k+1}}\right).$$

$$\boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} < 1 \;\Rightarrow\; \text{Gains of the additive approach.}$$

Numerical results

- 2D Poisson problem on Square - Dirichlet BC
- 3D Poisson problem on Cube - Dirichlet BC
- 2D Anisotropic Elliptic problem on Square - Dirichlet BC
- 2D Poisson problem on Annulus - Mixed BC

Numerical results

2D Poisson problem on Square - Dirichlet BC

3D Poisson problem on Cube - Dirichlet BC

2D Anisotropic Elliptic problem on Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Consider the Poisson problem in 2D with Dirichlet BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = (-0.5\pi, 0.5\pi)^2$ and $\mathcal{M} = [-0.5, 0.5]^2$ ($p = 2$ parameters).

Analytical solution: $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathcal{M}$

$$u(\boldsymbol{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{2}\right) \sin(2x) \sin(2y).$$

Parametric PINN training:

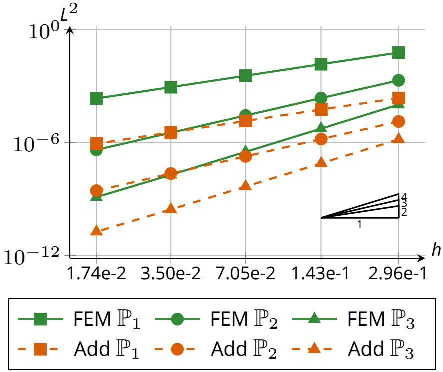
MLP 5 layers (sine - 40, 60, 60, 60, 40); LBFGs optimizer (5000 epochs - 6000 col. pts).
Imposing the Dirichlet BC exactly in the PINN with the levelset φ defined by

$$\varphi(\boldsymbol{x}) = (x + 0.5\pi)(x - 0.5\pi)(y + 0.5\pi)(y - 0.5\pi).$$

Online phase - 1 parameters instance

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22)$$

Error estimates in L^2 norm:



Online phase - 1 parameters instance

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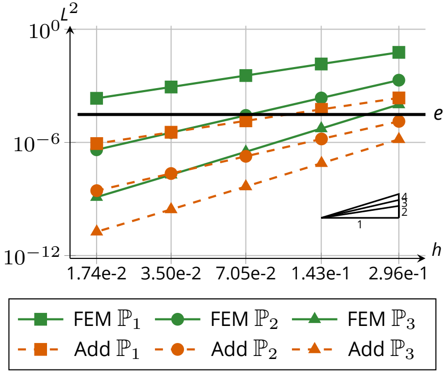


N_{dofs} required to reach the same error e:

k	e	N_{dofs} FEM	N_{dofs} Add
1	$1 \cdot 10^{-3}$	14 161	64
	$1 \cdot 10^{-4}$	143 641	576
2	$1 \cdot 10^{-4}$	6 889	225
	$1 \cdot 10^{-5}$	31 329	1 089
3	$1 \cdot 10^{-5}$	6 724	784
	$1 \cdot 10^{-6}$	20 164	2 704

$$h \approx N_{\text{dofs}}^{-1/2}$$

Numerical results

Problem considered

Problem statement: Consider the Poisson problem in 3D with Dirichlet BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = (-0.5\pi, 0.5\pi)^3$ and $\mathcal{M} = [-0.5, 0.5]^3$ ($p = 3$ parameters).

Analytical solution: $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathcal{M}$

$$u(\boldsymbol{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2 + (z-\mu_3)^2}{2}\right) \sin(2x)\sin(2y)\sin(2z).$$

Parametric PINN training:

MLP 5 layers (tanh - 40, 60, 60, 60, 40); LBFGs optimizer (5000 epochs - 40000 col. pts).

Imposing the Dirichlet BC exactly in the PINN with the levelset φ defined by

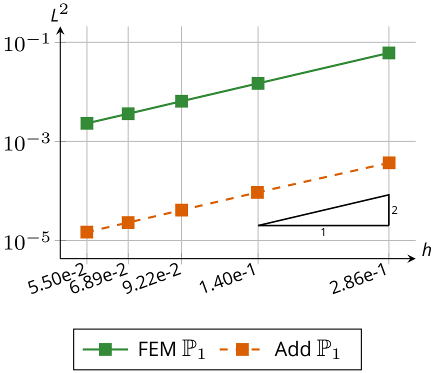
$$\varphi(\boldsymbol{x}) = (x + 0.5\pi)(x - 0.5\pi)(y + 0.5\pi)(y - 0.5\pi)(z + 0.5\pi)(z - 0.5\pi).$$

Training time: 12 minutes on NVIDIA RTX 2000 (8GB VRAM).

Online phase - 1 parameters instance I

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22, 0.1)$$

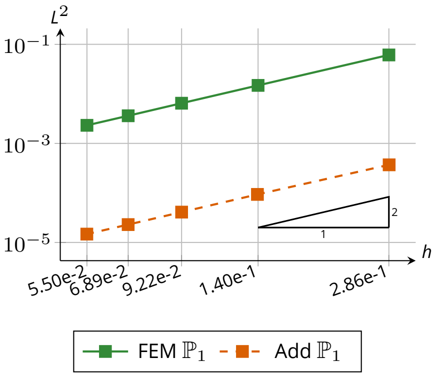
Error estimates in L^2 norm:



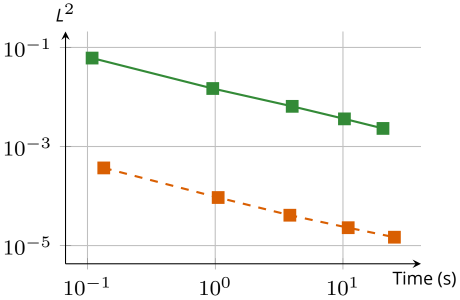
Online phase - 1 parameters instance I

$$\mu^{(1)} = (0.05, 0.22, 0.1)$$

Error estimates in L^2 norm:



L^2 error w.r.t. online time*:



* mesh + assembly + solve

+ prior derivatives evaluation

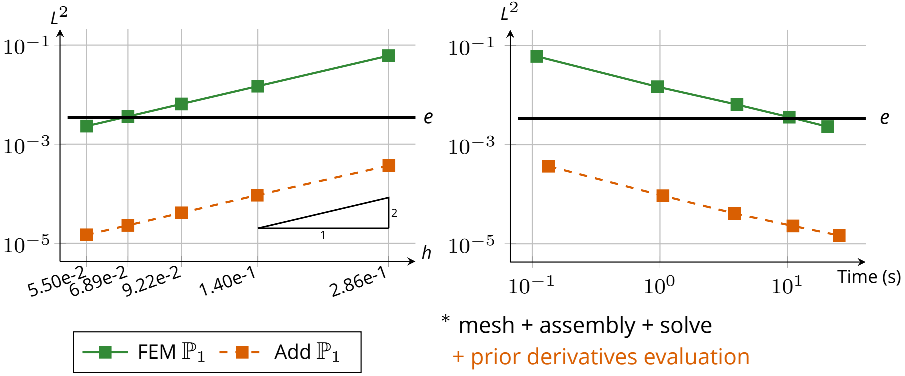
~~+ training time~~

Online phase - 1 parameters instance I

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22, 0.1)$$

Error estimates in L^2 norm:

L^2 error w.r.t. online time*:



* mesh + assembly + solve

+ prior derivatives evaluation

~~+ training time~~

Online phase - 1 parameters instance II

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22, 0.1)$$

N_{dofs} and online time required to reach the same error e:

	N		N_{dofs}		Online time*	
e	**FEM**	**Add**	**FEM**	**Add**	**FEM**	**Add**
$1 \cdot 10^{-3}$	152	12	$3.51 \cdot 10^6$	$1.73 \cdot 10^3$	$(7.65 \cdot 10^1)$	$5.2 \cdot 10^{-2}$
$1 \cdot 10^{-4}$	484	39	$1.13 \cdot 10^8$	$5.93 \cdot 10^4$	$(2.8 \cdot 10^3)$	$9.26 \cdot 10^{-1}$
$1 \cdot 10^{-5}$	1 539	122	$3.65 \cdot 10^9$	$1.82 \cdot 10^6$	$(1.02 \cdot 10^5)$	$(5.26 \cdot 10^1)$

$\div 1471$ →

Cartesian mesh: $N_{\text{dofs}} = N^3$ nodes ($\mathbb{P}_1$ elements).

* The results in brackets are extrapolations.

Online phase - 1 parameters instance II

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22, 0.1)$$

N_{dofs} and online time required to reach the same error e:

	N		N_{dofs}		Online time*	
e	**FEM**	**Add**	**FEM**	**Add**	**FEM**	**Add**
$1 \cdot 10^{-3}$	152	12	$3.51 \cdot 10^{6}$	$1.73 \cdot 10^{3}$	$(7.65 \cdot 10^{1})$	$5.2 \cdot 10^{-2}$
$1 \cdot 10^{-4}$	484	39	$1.13 \cdot 10^{8}$	$5.93 \cdot 10^{4}$	$(2.8 \cdot 10^{3})$	$9.26 \cdot 10^{-1}$
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$\div 1471$

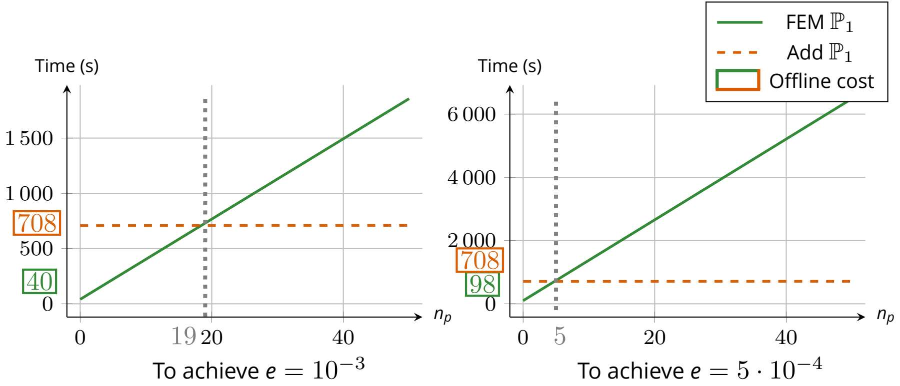
Cartesian mesh: $N_{\text{dofs}} = N^3$ nodes ($\mathbb{P}_1$ elements).

* The results in brackets are extrapolations.

⚠ Training time !

Parametric framework

How many parameters instance n_p are needed to balance training?



Online: assembly + solve + prior derivatives evaluation

Offline: mesh + training time

Numerical results

2D Poisson problem on Square - Dirichlet BC
3D Poisson problem on Cube - Dirichlet BC
2D Anisotropic Elliptic problem on Square - Dirichlet BC
2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Considering an Anisotropic Elliptic problem with Dirichlet BC:

$$\begin{cases} -\mathrm{div}(D\nabla u) = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = (0,1)^2$ and $\mathcal{M} = [0.4, 0.6]^2 \times [0.01, 1] \times [0.1, 0.8]$ ($p = 4$ parameters).

Right-hand side and diffusion matrix: $\boldsymbol{\mu} = (\mu_1, \mu_2, \sigma, \epsilon) \in \mathcal{M}$

$$f(\boldsymbol{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{0.025\sigma^2}\right).$$

$$D(\boldsymbol{x}, \boldsymbol{\mu}) = \begin{pmatrix} \epsilon x^2 + y^2 & (\epsilon - 1)xy \\ (\epsilon - 1)xy & x^2 + \epsilon y^2 \end{pmatrix}.$$

Parametric PINN training:

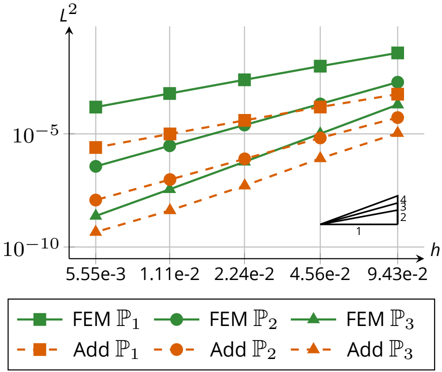
MLP[1] 5 layers (tanh - 40, 60, 60, 60, 40). Adam optimizer (15000 epochs - 8000 col. pts).
Imposing the Dirichlet BC exactly in the PINN with the a level-set function.

[1] with Fourier Features [Tancik, Srinivasan, and al, 2020]

Online phase

Error estimates: 1 parameter instance

$$\boldsymbol{\mu}^{(1)} = (0.51, 0.54, 0.52, 0.55)$$



Online phase

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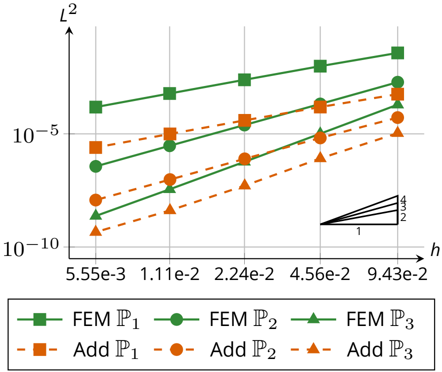


Gains achieved: $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

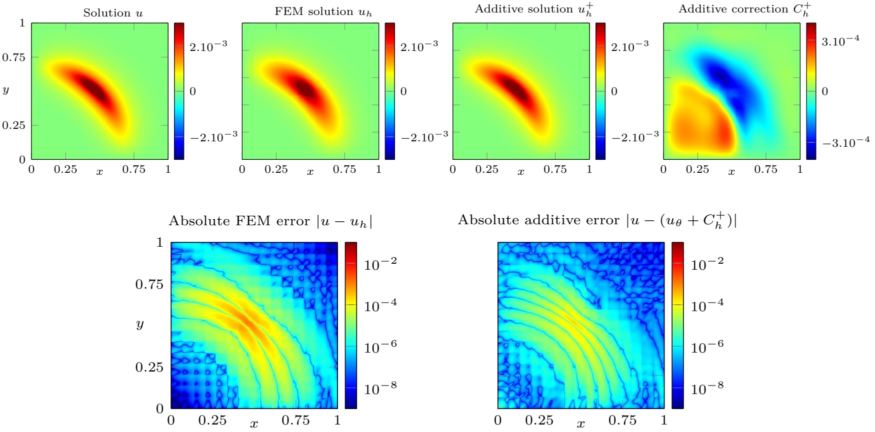
Considering $n_p = 50$ parameters instances

$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \ldots, \boldsymbol{\mu}^{(n_p)} \right\}.$$

	Gains in L^2 rel error of our method w.r.t. FEM		
k	**min**	**max**	**mean**
1	7.12	82.57	35.67
2	3.54	35.88	18.32
3	1.33	26.51	8.32

Cartesian mesh: N^2 nodes with $N = 20$.

Numerical solutions and errors



$$\boldsymbol{\mu}^{(2)} = (0.46, 0.52, 0.05, 0.12) \quad (k = 2, N = 16)$$

Numerical results

2D Poisson problem on Square - Dirichlet BC

3D Poisson problem on Cube - Dirichlet BC

2D Anisotropic Elliptic problem on Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Considering the Poisson problem with mixed BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = g, & \text{on } \Gamma_E \times \mathcal{M}, \\ \dfrac{\partial u}{\partial n} + u = g_R, & \text{on } \Gamma_I \times \mathcal{M}, \end{cases}$$

with $\Omega = \{(x,y) \in \mathbb{R}^2, 0.25 < x^2 + y^2 < 1\}$ and $\mathcal{M} = [2.4, 2.6]$ ($p = 1$ parameter).

Analytical solution and boundary conditions: $\boldsymbol{\mu} = \mu_1 \in \mathcal{M}$

$$u(\boldsymbol{x};\boldsymbol{\mu}) = 1 - \frac{\ln\left(\mu_1\sqrt{x^2+y^2}\right)}{\ln(4)},$$

$$g(\boldsymbol{x};\boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1)}{\ln(4)} \quad \text{and} \quad g_R(\boldsymbol{x};\boldsymbol{\mu}) = 2 + \frac{4-\ln(\mu_1)}{\ln(4)}.$$

Parametric PINN training:

MLP 5 layers (tanh - 5×40); LBFGs optimizer (4000 epochs - 6000 col. pts).

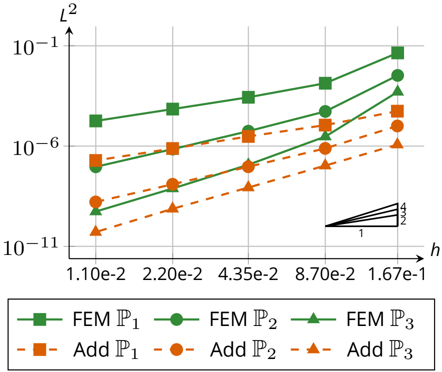
Imposing the mixed BC exactly in the PINN[1] (+ Sobolev training).

[1] [Sukumar and Srivastava, 2022]

Online phase

Error estimates: 1 parameters instance

$$\boldsymbol{\mu}^{(1)} = 2.51$$



Gains achieved: $\|u-u_h\|_{L^2}/\|u-u_h^+\|_{L^2}$

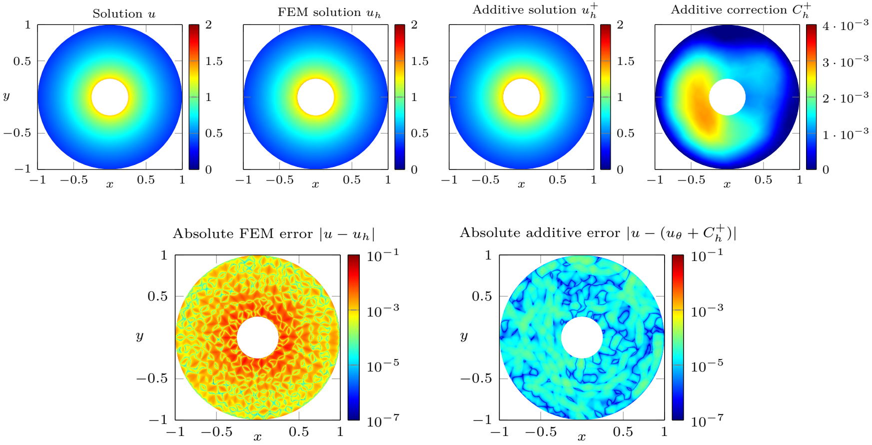
Considering $n_p = 50$ parameters instances

$$\mathcal{S} = \left\{\boldsymbol{\mu}^{(1)},\dots,\boldsymbol{\mu}^{(n_p)}\right\}.$$

	Gains in L^2 rel error of our method w.r.t. FEM		
k	**min**	**max**	**mean**
1	15.12	137.72	55.5
2	31	77.46	58.41
3	18.72	21.49	20.6

Mesh size: $h = 1.33 \cdot 10^{-1}$

Numerical solutions and errors



$$\mu^{(1)} = 2.51 \quad (k = 1, h = 1.67 \cdot 10^{-1})$$

Conclusion

- **Offline/online framework:**
  - **Offline phase:** Training a PINN over a parametric space.
  - **Online phase:** (fixed parameter instance)
    Resolution of the FEM correction problem using the PINN prior.
- **Numerical results:** Quick and accurate on several parametrics PDEs.
- **Theoretical results:** Convergence analysis showing the PINN prior can improve the FEM error.

Ongoing works:

- Extension to non-linear problems : First promising results for the incompressible Navier-Stokes equations. Appendix 1
- Extensions to DG methods and non-conforming FEM, currently under development in ScimBa (SciML's Python library) using auto-differentiation.

See the paper

References

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Thank you for your attention!

See the paper

Appendix 1 : Non-linear test case

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A1 – Heated cavity test case

Stationary incompressible Navier-Stokes equations (with buoyancy and gravity)[1] **:**
We consider $\Omega = [-1,1]^2$ a squared domain and $\boldsymbol{e}_y = (0,1)$.
Find the velocity $\boldsymbol{u} = (u_1, u_2)$, the pressure p and the temperature T such that

$$\begin{cases} (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p - \nu \Delta \boldsymbol{u} - g(\beta T + 1)\boldsymbol{e}_y = 0 \ \text{ in } \Omega & \text{(momentum)} \\ \nabla \cdot \boldsymbol{u} = 0 \ \text{ in } \Omega & \text{(incompressibility)} \\ \boldsymbol{u} \cdot \nabla T - k_f \Delta T = 0 \ \text{ in } \Omega & \text{(energy)} \\ + \text{ suitable BC} \end{cases} \qquad (\mathcal{P})$$

with $g = 9.81$ the gravity, $\beta = 0.1$ the expansion coefficient, ν the viscosity and k_f the thermal conductivity. [Coulaud, Le, and Duvigneau, 2024]

[1] **[\textcolor {darkred}{The approach can be extended to other test cases.}]**

A1 – Heated cavity test case

Objective: Simulation on a range of parameters $\boldsymbol{\mu} = (\nu, k_f) \in \mathcal{M} = [0.01, 0.1]^2$.

Stationary incompressible Navier-Stokes equations (with buoyancy and gravity) :

We consider $\boldsymbol{x} = (x, y) \in \Omega$ and $\boldsymbol{e}_y = (0, 1)$.

Find $\boldsymbol{U} = (\boldsymbol{u}, p, T) = (u_1, u_2, p, T)$ such that

$$\begin{cases} R_{mom}(U; \boldsymbol{x}, \boldsymbol{\mu}) = 0 \quad \text{in } \Omega & \text{(momentum)} \\ R_{inc}(U; \boldsymbol{x}, \boldsymbol{\mu}) = 0 \quad \text{in } \Omega & \text{(incompressibility)} \\ R_{ener}(U; \boldsymbol{x}, \boldsymbol{\mu}) = 0 \quad \text{in } \Omega & \text{(energy)} \\ + \text{ suitable BC} \end{cases} \qquad (\mathcal{P})$$

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Boundary Conditions:

No-slip BC : $\boldsymbol{u} = 0$ on $\partial\Omega$

Isothermal BC : $T = 1$ on the left wall ($x = -1$)
$T = -1$ on the right wall ($x = 1$)

Adiabatic BC : $\dfrac{\partial T}{\partial n} = 0$ on the top and bottom walls ($y = \pm 1$, denoted by Γ_{ad})

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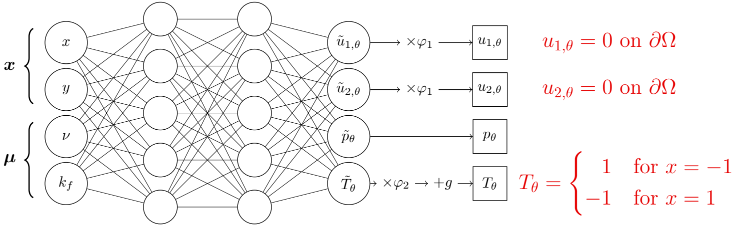
A1.1 – Neural Network considered

We consider a parametric NN with 4 inputs and 4 outputs, defined by

$$U_\theta(\boldsymbol{x}, \boldsymbol{\mu}) = \big(u_{1,\theta}, u_{2,\theta}, p_\theta, T_\theta\big)(\boldsymbol{x}, \boldsymbol{\mu}).$$

The Dirichlet boundary conditions are imposed on the outputs of the MLP by a **post-processing** step. [Sukumar and Srivastava, 2022]



We consider two levelsets functions φ_1 and φ_2, and the linear function g defined by

$$\varphi_1(x, y) = (x-1)(x+1)(y-1)(y+1),$$

$$\varphi_2(x, y) = (x-1)(x+1) \quad \text{and} \quad g(x, y) = 1 - (x+1).$$

A1.1 – PINN training

Approximate the solution of $(\mathcal{P})$ **by a PINN :** Find the optimal weights $\theta^\star$, such that

$$\theta^\star = \underset{\theta}{\mathrm{argmin}} \left(J_{inc}(\theta) + J_{mom}(\theta) + J_{ener}(\theta) + J_{ad}(\theta) \right), \qquad (\mathcal{P}_\theta)$$

where the different cost functions[1] are defined by

adiabatic condition

$$J_{ad}(\theta) = \int_{\mathcal{M}} \int_{\Gamma_{ad}} \left| \frac{\partial T_\theta(\boldsymbol{x},\boldsymbol{\mu})}{\partial n} \right|^2 d\boldsymbol{x} d\boldsymbol{\mu},$$

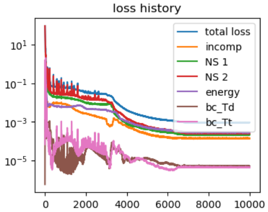
3 residual losses

$$J_\bullet(\theta) = \int_{\mathcal{M}} \int_{\Omega} \left| R_\bullet(U_\theta(\boldsymbol{x},\boldsymbol{\mu}); \boldsymbol{x}, \boldsymbol{\mu}) \right|^2 d\boldsymbol{x} d\boldsymbol{\mu},$$

with U_θ the parametric NN and • the PDE considered (i.e. *inc*, *mom* or *ener*).

Network - MLP	
layers	40, 60, 60, 60, 40
σ	sine

Training (ADAM / LBFGs)			
lr	7e-3	N_{col}	40000
n_{epochs}	10000	N_{bc}	30000

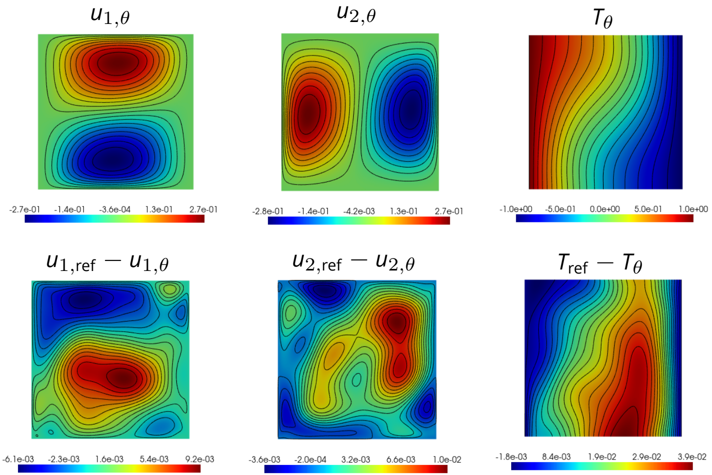


[1] [**Discretized by a random process using Monte-Carlo method.**]

A1.1 – Prediction on $\mu^{(1)} = (0.1, 0.1)$

Prediction :

Error map :



L^2 error : (relative)	2.98×10^{-2}	3.17×10^{-2}	3.90×10^{-2}

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A1.2 – Enriched space using PINN

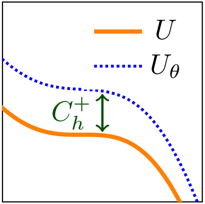
Considering the PINN prior $U_\theta = (\boldsymbol{u}_\theta, p_\theta, T_\theta)$, we define the mixed finite element space additively enriched by the PINN as follows:

$$M_h^+ = \left\{ U_h^+ = U_\theta + C_h^+, \quad C_h^+ \in M_h^0 \right\}$$

with $M_h^0 = [V_h^0]^2 \times Q_h \times W_h^0$, $U_h^+ = (\boldsymbol{u}_h^+, p_h^+, T_h^+) \in M_h^+$ and $C_h^+ = (\boldsymbol{C}_{h,\boldsymbol{u}}^+, C_{h,p}^+, C_{h,T}^+)$.

We can then define the three finite element subspaces of M_h^+ as follows:

$$\begin{aligned} \boldsymbol{V}_h^+ &= \left\{ \boldsymbol{u}_h^+ = \boldsymbol{u}_\theta + \boldsymbol{C}_{h,\boldsymbol{u}}^+, \; \boldsymbol{C}_{h,\boldsymbol{u}}^+ \in [V_h^0]^2 \right\}, \\ Q_h^+ &= \left\{ p_h^+ = p_\theta + C_{h,p}^+, \; C_{h,p}^+ \in Q_h \right\}, \\ W_h^+ &= \left\{ T_h^+ = T_\theta + C_{h,T}^+, \; C_{h,T}^+ \in W_h^0 \right\}, \end{aligned}$$



where $\boldsymbol{C}_{h,\boldsymbol{u}}^+$, $C_{h,p}^+$ and $C_{h,T}^+$ becomes the unknowns.

A1.2 – Weak form - Additive approach

Weak problem[1] **:** Find $C_h^+ = (C_{h,\boldsymbol{u}}^+, C_{h,p}^+, C_{h,T}^+) \in M_h^0$ s.t., $\forall (\boldsymbol{v}_h, q_h, w_h) \in M_h^0$,

$$
\begin{aligned}
&\int_\Omega \left[(\boldsymbol{u}_\theta \cdot \nabla)\boldsymbol{u}_\theta + (\boldsymbol{u}_\theta \cdot \nabla)\boldsymbol{C}_{h,\boldsymbol{u}}^+ + (\boldsymbol{C}_{h,\boldsymbol{u}}^+ \cdot \nabla)\boldsymbol{u}_\theta + (\boldsymbol{C}_{h,\boldsymbol{u}}^+ \cdot \nabla)\boldsymbol{C}_{h,\boldsymbol{u}}^+\right] \cdot \boldsymbol{v_h}\, d\boldsymbol{x} \\
&\qquad + \mu \left(\int_\Omega \nabla \boldsymbol{u}_\theta : \nabla \boldsymbol{v}_h\, d\boldsymbol{x} + \int_\Omega \nabla \boldsymbol{C}_{h,\boldsymbol{u}}^+ : \nabla \boldsymbol{v}_h\, d\boldsymbol{x}\right) + \left(\int_\Omega \nabla p_\theta \cdot \boldsymbol{v}_h\, d\boldsymbol{x} - \int_\Omega C_{h,p}^+ \nabla \cdot \boldsymbol{v}_h\, d\boldsymbol{x}\right) \\
&\qquad\qquad - g \int_\Omega (1 + \beta(T_\theta + C_{h,T}^+))\boldsymbol{e}_y \cdot \boldsymbol{v}_h\, d\boldsymbol{x} = 0, \ \text{(momentum)} \\
&\int_\Omega q_h \left[\nabla \cdot \boldsymbol{u}_\theta + \nabla \cdot \boldsymbol{C}_{h,\boldsymbol{u}}^+\right] d\boldsymbol{x} \ + \ 10^{-4} \int_\Omega q_h \left(p_\theta + C_{h,p}^+\right) d\boldsymbol{x} = 0, \ \text{(incompressibility + penal)} \\
&\int_\Omega \left[\boldsymbol{u}_\theta \cdot \nabla T_\theta + \boldsymbol{u}_\theta \cdot \nabla C_{h,T}^+ + \boldsymbol{C}_{h,\boldsymbol{u}}^+ \cdot \nabla T_\theta + \boldsymbol{C}_{h,\boldsymbol{u}}^+ \cdot \nabla C_{h,T}^+\right] w_h\, d\boldsymbol{x} \\
&\qquad + k_f \left(\int_\Omega \nabla T_\theta \cdot \nabla w_h\, d\boldsymbol{x} + \int_\Omega \nabla C_{h,T}^+ \cdot \nabla w_h\, d\boldsymbol{x}\, w_h\, d\boldsymbol{s}\right) = 0, \ \text{(energy)}
\end{aligned}
\qquad (\mathcal{P}_h^+)
$$

with $U_\theta = (\boldsymbol{u}_\theta, p_\theta, T_\theta)$ the PINN prior and some modified boundary conditions.

[1] [**Problem solved using Newton's algorithm with the correction vector initialized to 0.**]

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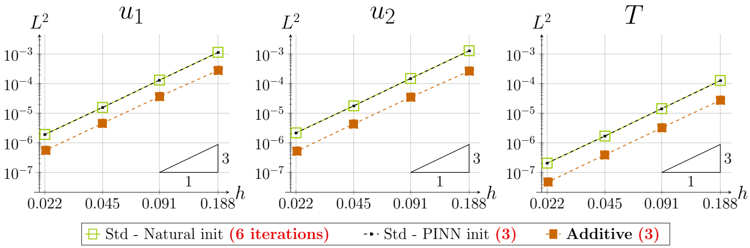
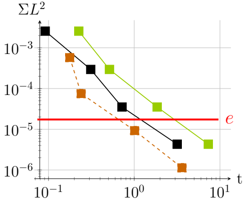
A1.3 – Numerical costs

$\boldsymbol{\mu}^{(1)}$:



N_{dofs} and execution time required to reach the same global L^2 relative error e :[1]



Std vs Add	Number of DoFs		Execution times		
e	Std	Add	(nat)	(PINN)	Add
$1 \cdot 10^{-3}$	6,031	2,044	0.32	0.16	0.16
$1 \cdot 10^{-4}$	26,959	10,588	0.99	0.48	0.23
$1 \cdot 10^{-5}$	121,156	49,231	4.21	1.75	0.96

÷2.5 ÷2 ÷4

[1] [Error defined as the sum of the L^2 relatives errors on $\bm {u}$ and T.]