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Enriching continuous Lagrange finite element approximation spaces using neural networks

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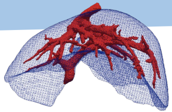
MS : "Perspectives et avancées récentes en apprentissage pour la résolution numérique d'équations aux dérivées partielles"

CANUM 2026

Scientific context



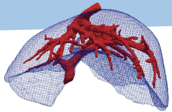
Context: Create real-time digital twins of an organ (e.g. liver).

Objective: Develop an hybrid finite element / neural network method.

finite element: accurate

neural network: quick + parameterized

Scientific context



Context: Create real-time digital twins of an organ (e.g. liver).

Objective: Develop an hybrid finite element / neural network method.

accurate — quick + parameterized

Parametric Poisson problem (with homogeneous Dirichlet BC):

For one or several $\boldsymbol{\mu} \in \mathcal{M}$, find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u(\boldsymbol{x};\boldsymbol{\mu}) = f(\boldsymbol{x};\boldsymbol{\mu}), & (\boldsymbol{x},\boldsymbol{\mu}) \in \Omega \times \mathcal{M}, \\ u(\boldsymbol{x};\boldsymbol{\mu}) = 0, & (\boldsymbol{x},\boldsymbol{\mu}) \in \partial\Omega \times \mathcal{M}, \end{cases} \qquad (\mathcal{P})$$

with $\Omega \subset \mathbb{R}^d$ a domain (d the spatial dimension).

Classical approaches

Physics-Informed Neural Networks

Standard PINNs: Find the optimal weights $\theta^\star$, such that

$$\theta^\star = \underset{\theta}{\operatorname{argmin}} \left(\omega_r J_r(\theta) + \omega_b J_b(\theta)\right), \qquad (\mathcal{P}_\theta)$$

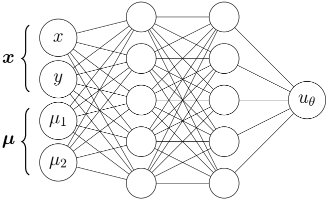
with

residual loss
$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} \left|\Delta u_\theta(\boldsymbol{x}, \boldsymbol{\mu}) + f(\boldsymbol{x}, \boldsymbol{\mu})\right|^2 d\boldsymbol{x} d\boldsymbol{\mu},$$

boundary loss
$$J_b(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega} \left|u_\theta(\boldsymbol{x}, \boldsymbol{\mu})\right|^2 d\boldsymbol{x} d\boldsymbol{\mu},$$

with ω_r and ω_b are some weights.

u_θ is a neural network, e.g. fully-connected NN.
(see example in 2D with 2 parameters)



Monte-Carlo method: Discretize the cost functions by random process.

Finite Element Method[1]

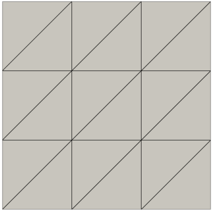
Variational Problem:

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \qquad (\mathcal{P}_h)$$

with h the characteristic mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h, \quad l(v_h) = \int_\Omega f v_h,$$

and V_h^0 a continuous, piecewise polynomial space of degree k.



Cartesian mesh

[1] [Ern and Guermond, 2004]

Finite Element Method[1]

Variational Problem:

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with *h* the characteristic mesh size, *a* and *l* the bilinear and linear forms given by

$$a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h, \quad l(v_h) = \int_\Omega f v_h,$$

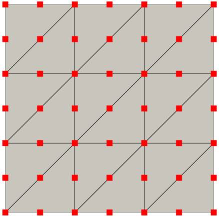
and V_h^0 a continuous, piecewise polynomial space of degree k.

Linear system: Let $(\phi_1, \dots, \phi_{N_{\text{dofs}}})$ a basis of V_h^0.
Find $U \in \mathbb{R}^{N_{\text{dofs}}}$ such that

$$AU = b,$$

$$A = \big(a(\phi_i, \phi_j)\big)_{1 \le i,j \le N_{\text{dofs}}} \in \mathbb{R}^{N_{\text{dofs}} \times N_{\text{dofs}}},$$

$$b = \big(l(\phi_j)\big)_{1 \le j \le N_{\text{dofs}}} \in \mathbb{R}^{N_{\text{dofs}}}.$$



degrees of freedom $(k = 2)$

[1] [Ern and Guermond, 2004]

Finite Element Method[1]

Variational Problem:

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \qquad (\mathcal{P}_h)$$

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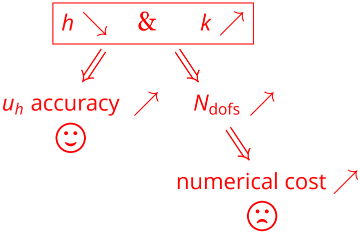
$$b = \left(l(\phi_j)\right)_{1 \le j \le N_{\text{dofs}}} \in \mathbb{R}^{N_{\text{dofs}}}.$$



[1] [Ern and Guermond, 2004]

Enriched finite element method using PINNs

Pipeline of the Enriched FEM

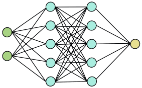
OFFLINE : PINN training

Inputs

$\boldsymbol{x} \in \Omega$: space coordinates

$\boldsymbol{\mu} \in \mathcal{M}$: parameter

$\longrightarrow$



Parametric PINN

$\longrightarrow$

Output

$u_\theta(\boldsymbol{x}, \boldsymbol{\mu})$: prediction of the PDE solution $u(\boldsymbol{x}, \boldsymbol{\mu})$

Pipeline of the Enriched FEM

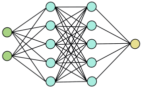
OFFLINE : PINN training

Inputs

$\boldsymbol{x} \in \Omega$: space coordinates

$\boldsymbol{\mu} \in \mathcal{M}$: parameter

→



Parametric PINN

→

Output

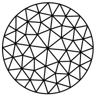
$u_\theta(\boldsymbol{x}, \boldsymbol{\mu})$: prediction of the PDE solution $u(\boldsymbol{x}, \boldsymbol{\mu})$

ONLINE (fixed $\boldsymbol{\mu}$) : PINN evaluation + Enriched FEM resolution

PINN evaluation

$u_\theta(\cdot, \boldsymbol{\mu})$: prediction for the given parameter μ

→



FEM solver enriched by u_θ

→

Output

enriched FEM solution (depending on the mesh size h)

Complete ONLINE process : quick + accurate

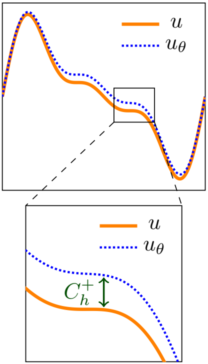
Additive approach

Variational Problem: Let $u_\theta \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ be a PINN prior.

$$\text{Find } C_h^+ \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(C_h^+, v_h) = l(v_h) - a(u_\theta, v_h), \qquad (\mathcal{P}_h^+)$$

with the enriched trial space V_h^+ defined by

$$V_h^+ = \left\{ u_h^+ = u_\theta + C_h^+, \quad C_h^+ \in V_h^0 \right\}.$$



General Dirichlet BC: If $u = g$ on $\partial\Omega$, then

$$C_h^+ = g - u_\theta \quad \text{on } \partial\Omega,$$

with u_θ the PINN prior.

We expect that the modified problem will give the same results as the standard one on coarser meshes.

Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

We denote $u_h \in V_h$ the solution of $(\mathcal{P}_h)$ with V_h the standard trial space. Then,

$$\|u - u_h\|_{L^2} \leqslant C_{L^2}\, h^{k+1} |u|_{H^{k+1}}\,.$$

Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

We denote $u_h \in V_h$ the solution of $(\mathcal{P}_h)$ with V_h the standard trial space. Then,

$$\|u - u_h\|_{L^2} \leqslant C_{L^2}\, h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [Lecourtier et al., 2026]

We denote $u_h^+ \in V_h^+$ the solution of $(\mathcal{P}_h^+)$ with V_h^+ the enriched trial space. Then,

$$\|u - u_h^+\|_{L^2} \leqslant \boxed{\frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} \left(C_{L^2}\, h^{k+1} |u|_{H^{k+1}} \right).$$

$$\boxed{\frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} < 1 \Rightarrow \text{Gains of the additive approach.}$$

Numerical results

- 2D Poisson problem on Square - Dirichlet BC
- 3D Poisson problem on Cube - Dirichlet BC
- 2D Anisotropic Elliptic problem on Square - Dirichlet BC
- 2D Poisson problem on Annulus - Mixed BC

Numerical results

2D Poisson problem on Square - Dirichlet BC

3D Poisson problem on Cube - Dirichlet BC

2D Anisotropic Elliptic problem on Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Consider the Poisson problem in 2D with Dirichlet BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = (-0.5\pi, 0.5\pi)^2$ and $\mathcal{M} = [-0.5, 0.5]^2$ ($p = 2$ parameters).

Analytical solution: $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathcal{M}$

$$u(\boldsymbol{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{2}\right) \sin(2x)\sin(2y).$$

Parametric PINN training:

MLP 5 layers (sine - 40, 60, 60, 60, 40); LBFGs optimizer (5000 epochs - 6000 col. pts).

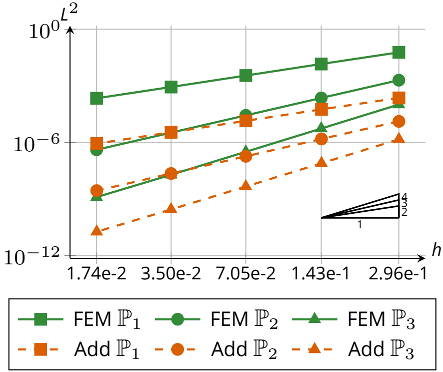
Imposing the Dirichlet BC exactly in the PINN with the levelset φ defined by

$$\varphi(\boldsymbol{x}) = (x + 0.5\pi)(x - 0.5\pi)(y + 0.5\pi)(y - 0.5\pi).$$

Online phase - 1 parameters instance

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22)$$

Error estimates in L^2 norm:



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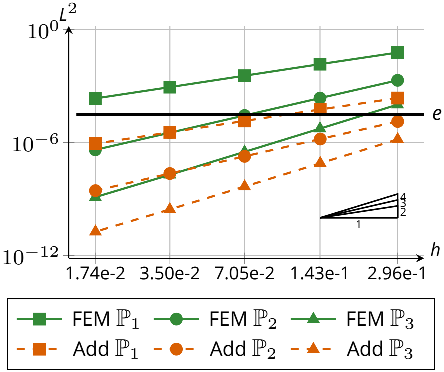


N_{dofs} required to reach the same error e:

		N_{dofs}	
k	**e**	**FEM**	**Add**
1	$1 \cdot 10^{-3}$	14 161	64
	$1 \cdot 10^{-4}$	143 641	576
2	$1 \cdot 10^{-4}$	6 889	225
	$1 \cdot 10^{-5}$	31 329	1 089
3	$1 \cdot 10^{-5}$	6 724	784
	$1 \cdot 10^{-6}$	20 164	2 704

$$h \approx N_{\text{dofs}}^{-1/2}$$

Numerical results

2D Poisson problem on Square - Dirichlet BC

3D Poisson problem on Cube - Dirichlet BC

2D Anisotropic Elliptic problem on Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Consider the Poisson problem in 3D with Dirichlet BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = (-0.5\pi, 0.5\pi)^3$ and $\mathcal{M} = [-0.5, 0.5]^3$ ($p = 3$ parameters).

Analytical solution: $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathcal{M}$

$$u(\boldsymbol{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2 + (z-\mu_3)^2}{2}\right) \sin(2x)\sin(2y)\sin(2z).$$

Parametric PINN training:

MLP 5 layers (tanh - 40, 60, 60, 60, 40); LBFGs optimizer (5000 epochs - 40000 col. pts).
Imposing the Dirichlet BC exactly in the PINN with the levelset φ defined by

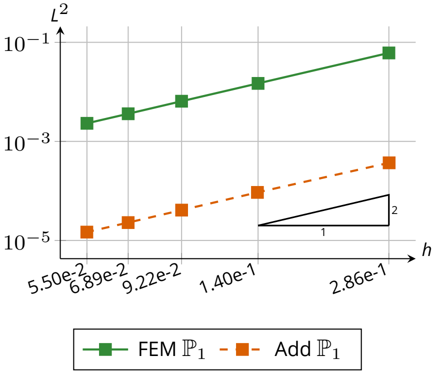
$$\varphi(\boldsymbol{x}) = (x+0.5\pi)(x-0.5\pi)(y+0.5\pi)(y-0.5\pi)(z+0.5\pi)(z-0.5\pi).$$

Training time: 12 minutes on NVIDIA RTX 2000 (8GB VRAM).

Online phase - 1 parameters instance I

$$\mu^{(1)} = (0.05, 0.22, 0.1)$$

Error estimates in L^2 norm:

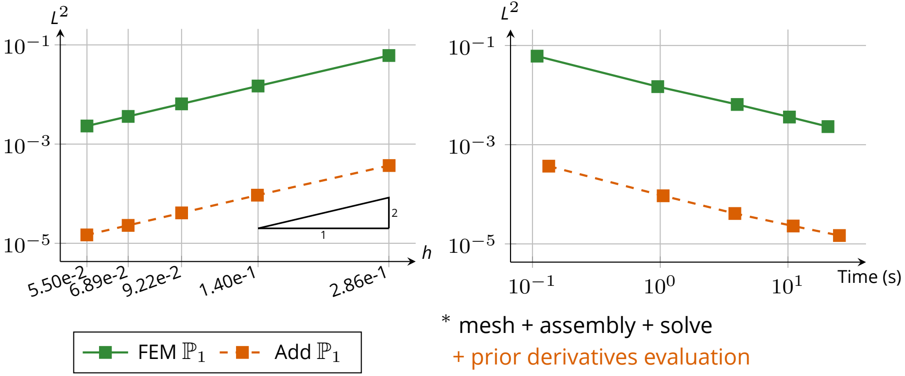


Online phase - 1 parameters instance I

$$\mu^{(1)} = (0.05, 0.22, 0.1)$$

Error estimates in L^2 norm:

L^2 error w.r.t. online time*:



* mesh + assembly + solve

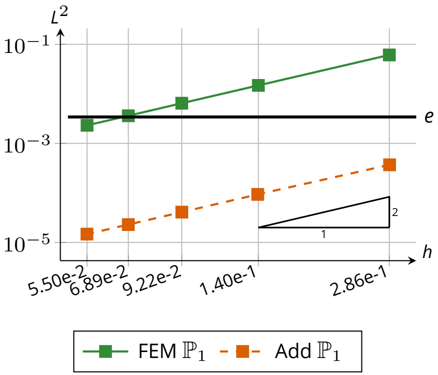
+ prior derivatives evaluation

~~+ training time~~

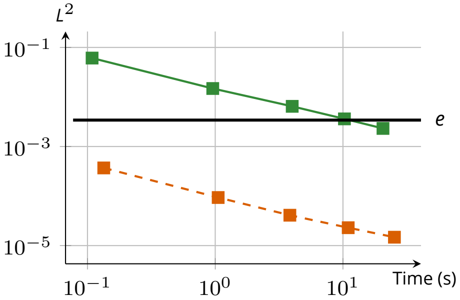
Online phase - 1 parameters instance I

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22, 0.1)$$

Error estimates in L^2 norm:



L^2 error w.r.t. online time*:



* mesh + assembly + solve
+ prior derivatives evaluation
~~+ training time~~

Online phase - 1 parameters instance II

$$\mu^{(1)} = (0.05, 0.22, 0.1)$$

N_{dofs} and online time required to reach the same error e:

	N		N_{dofs}		Online time*	
e	**FEM**	**Add**	**FEM**	**Add**	**FEM**	**Add**
$1 \cdot 10^{-3}$	152	12	$3.51 \cdot 10^{6}$	$1.73 \cdot 10^{3}$	$(7.65 \cdot 10^{1})$	$5.2 \cdot 10^{-2}$
$1 \cdot 10^{-4}$	484	39	$1.13 \cdot 10^{8}$	$5.93 \cdot 10^{4}$	$(2.8 \cdot 10^{3})$	$9.26 \cdot 10^{-1}$
$1 \cdot 10^{-5}$	1 539	122	$3.65 \cdot 10^{9}$	$1.82 \cdot 10^{6}$	$(1.02 \cdot 10^{5})$	$(5.26 \cdot 10^{1})$

$\div 1471$ →

Cartesian mesh: $N_{dofs} = N^3$ nodes ($\mathbb{P}_1$ elements).

* The results in brackets are extrapolations.

Online phase - 1 parameters instance II

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22, 0.1)$$

N_{dofs} and online time required to reach the same error e:

	N		N_{dofs}		Online time*	
e	**FEM**	**Add**	**FEM**	**Add**	**FEM**	**Add**
$1 \cdot 10^{-3}$	152	12	$3.51 \cdot 10^6$	$1.73 \cdot 10^3$	$(7.65 \cdot 10^1)$	$5.2 \cdot 10^{-2}$
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$\div 1471$

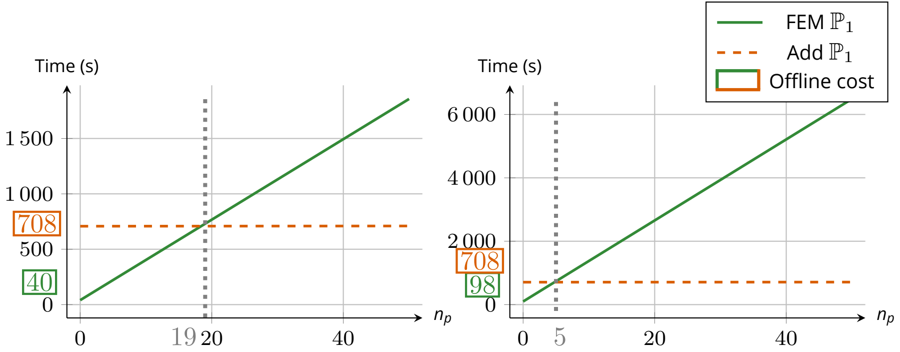
Cartesian mesh: $N_{\text{dofs}} = N^3$ nodes ($\mathbb{P}_1$ elements).

* The results in brackets are extrapolations.

⚠ Training time !

Parametric framework

How many parameters instance n_p are needed to balance training?



To achieve $e = 10^{-3}$ | To achieve $e = 5 \cdot 10^{-4}$

Online: assembly + solve + prior derivatives evaluation

Offline: mesh + training time

Numerical results

Problem considered

Problem statement: Considering an Anisotropic Elliptic problem with Dirichlet BC:

$$\begin{cases} -\mathrm{div}(D\nabla u) = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = (0,1)^2$ and $\mathcal{M} = [0.4, 0.6]^2 \times [0.01, 1] \times [0.1, 0.8]$ ($p = 4$ parameters).

Right-hand side and diffusion matrix: $\boldsymbol{\mu} = (\mu_1, \mu_2, \sigma, \epsilon) \in \mathcal{M}$

$$f(\boldsymbol{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{0.025\sigma^2}\right).$$

$$D(\boldsymbol{x}, \boldsymbol{\mu}) = \begin{pmatrix} \epsilon x^2 + y^2 & (\epsilon - 1)xy \\ (\epsilon - 1)xy & x^2 + \epsilon y^2 \end{pmatrix}.$$

Parametric PINN training:

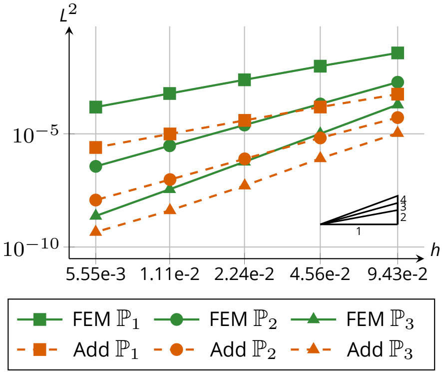
MLP[1] 5 layers (tanh - 40, 60, 60, 60, 40). Adam optimizer (15000 epochs - 8000 col. pts).
Imposing the Dirichlet BC exactly in the PINN with the a level-set function.

[1] with Fourier Features, Tancik et al. [2020]

Online phase

Error estimates: 1 parameter instance

$$\boldsymbol{\mu}^{(1)} = (0.51, 0.54, 0.52, 0.55)$$



Online phase

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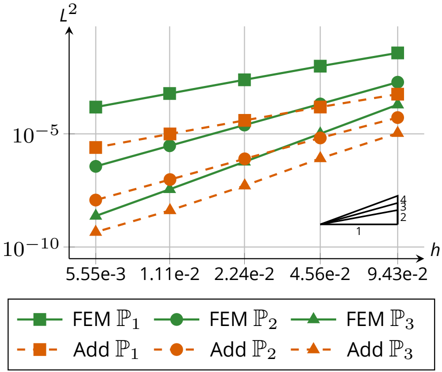


Gains achieved: $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

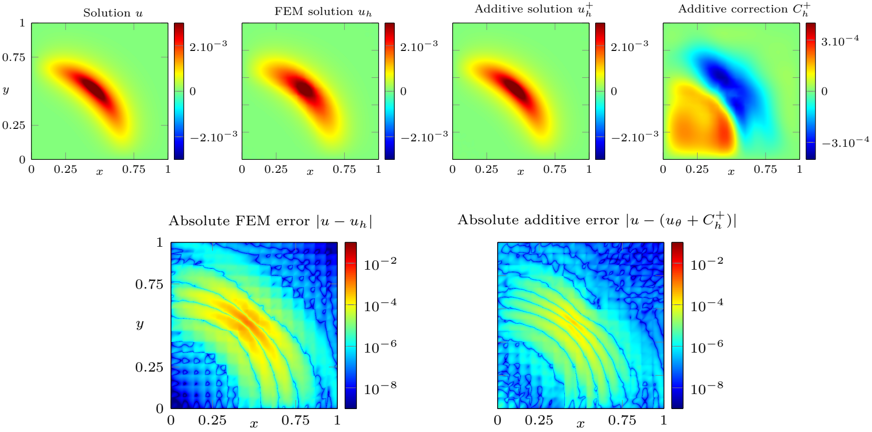
Considering $n_p = 50$ parameters instances

$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(n_p)} \right\}.$$

	Gains in L^2 rel error of our method w.r.t. FEM		
k	**min**	**max**	**mean**
1	7.12	82.57	35.67
2	3.54	35.88	18.32
3	1.33	26.51	8.32

Cartesian mesh: N^2 nodes with $N = 20$.

Numerical solutions and errors



$$\boldsymbol{\mu}^{(2)} = (0.46, 0.52, 0.05, 0.12) \quad (k = 2, N = 16)$$

Numerical results

2D Poisson problem on Square - Dirichlet BC

3D Poisson problem on Cube - Dirichlet BC

2D Anisotropic Elliptic problem on Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Considering the Poisson problem with mixed BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = g, & \text{on } \Gamma_E \times \mathcal{M}, \\ \dfrac{\partial u}{\partial n} + u = g_R, & \text{on } \Gamma_I \times \mathcal{M}, \end{cases}$$

with $\Omega = \{(x,y) \in \mathbb{R}^2, 0.25 < x^2 + y^2 < 1\}$ and $\mathcal{M} = [2.4, 2.6]$ ($p = 1$ parameter).

Analytical solution and boundary conditions: $\boldsymbol{\mu} = \mu_1 \in \mathcal{M}$

$$u(\boldsymbol{x};\boldsymbol{\mu}) = 1 - \frac{\ln\left(\mu_1 \sqrt{x^2+y^2}\right)}{\ln(4)},$$

$$g(\boldsymbol{x};\boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1)}{\ln(4)} \quad \text{and} \quad g_R(\boldsymbol{x};\boldsymbol{\mu}) = 2 + \frac{4 - \ln(\mu_1)}{\ln(4)}.$$

Parametric PINN training:

MLP 5 layers (tanh - 5×40); LBFGs optimizer (4000 epochs - 6000 col. pts).

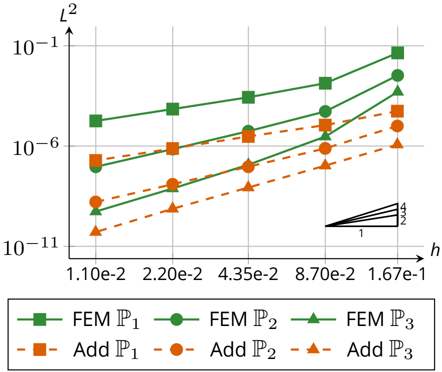
Imposing the mixed BC exactly in the PINN[1] (+ Sobolev training).

[1] [Sukumar and Srivastava, 2022]

Online phase

Error estimates: 1 parameters instance

$$\boldsymbol{\mu}^{(1)} = 2.51$$



Gains achieved: $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

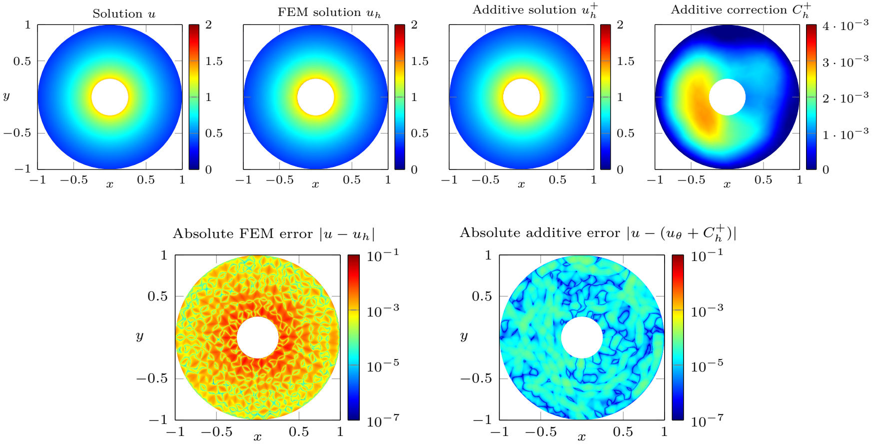
Considering $n_p = 50$ parameters instances

$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \ldots, \boldsymbol{\mu}^{(n_p)} \right\}.$$

	Gains in L^2 rel error of our method w.r.t. FEM		
k	**min**	**max**	**mean**
1	15.12	137.72	55.5
2	31	77.46	58.41
3	18.72	21.49	20.6

Mesh size: $h = 1.33 \cdot 10^{-1}$

Numerical solutions and errors



$$\mu^{(1)} = 2.51 \quad (k = 1, h = 1.67 \cdot 10^{-1})$$

Conclusion

- **Offline/online framework:**
  - **Offline phase:** Training a PINN over a parametric space.
  - **Online phase:** (fixed parameter instance)
    Quick and accurate resolution of the FEM correction problem using the PINN prior.
- **Numerical results:** Several parametrics PDEs.

Some extensions:

- Extension to non-linear problems : First promising results for the incompressible Navier-Stokes equations.
- Extensions to DG methods and non-conforming FEM, currently under development in ScimBa (SciML's Python library) using auto-differentiation

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Thank you for your attention!

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